Lecture 17:

Exponential Distributions

1° Def 17.1. A random variable T is said to have an exponential distribution with rate λ, or $T = \text{exponential}(\lambda)$, if

$$\mathbb{P}(T \leq t) = 1 - e^{-\lambda t}, \qquad \forall\, t \geq 0.$$

Remark 17.1. We describe the distribution by giving the cumulative distribution function (CDF) $F(t) = \mathbb{P}(T \leq t)$. The information given by F is equivalent to that encoded in the probability density function (PDF) $f_T(t)$, which is the derivative of the CDF:

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0; \\ 0, & t < 0. \end{cases}$$

Q: What's $\mathbb{E}T$? $\mathbb{E}T^2$? $var(T)$?

A: $$\mathbb{E}T = \int_0^\infty t \cdot f_T(t)\, dt$$

$$= \int_0^\infty t \cdot \lambda e^{-\lambda t}\, dt$$

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$$= t\cdot(-e^{-\lambda t})\Big|_0^\infty - \int_0^\infty 1\cdot(-e^{-\lambda t})\,dt$$

$$= 0 + \int_0^\infty e^{-\lambda t}\,dt$$

$$= \frac{1}{-\lambda}\cdot e^{-\lambda t}\Big|_0^\infty$$

$$= \frac{1}{\lambda}.$$

$$\mathbb{E}T^2 = \int_0^\infty t^2\cdot\lambda e^{-\lambda t}\,dt$$

$$= t^2\cdot(-e^{-\lambda t})\Big|_0^\infty - \int_0^\infty 2t\cdot(-e^{-\lambda t})\,dt$$

$$= 0 + 2\int_0^\infty t\,e^{-\lambda t}\,dt$$

$$= 2\cdot\frac{1}{\lambda}\cdot\mathbb{E}T$$

$$= \frac{2}{\lambda^2}.$$

$$\mathrm{Var}(T) = \mathbb{E}[(T-\mathbb{E}T)^2] = \mathbb{E}T^2 - (\mathbb{E}T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Remark 17.2. Let $S = \text{exponential}(1)$, $T = \text{exponential}(\lambda)$, then

$$\mathbb{P}(S/\lambda \le t) = \mathbb{P}(S \le \lambda t) = 1 - e^{-\lambda t} = \mathbb{P}(T \le t).$$

That is, $T = S/\lambda$.

This coincides with the fact that $\mathbb{E}[cX] = c\,\mathbb{E}[X]$,

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and that $\mathrm{var}(cX) = c^2 \mathrm{var}(X)$.

Notice that $\mathbb{E}[S] = 1$, and $\mathrm{Var}[S] = 1$.

$\mathbb{E}[S/\lambda] = \frac{1}{\lambda}\mathbb{E}[S] = \frac{1}{\lambda} = \mathbb{E}[T]$,

and $\mathrm{Var}[S/\lambda] = \frac{1}{\lambda^2}\mathrm{Var}[S] = \frac{1}{\lambda^2} = \mathrm{Var}(T)$.

2°. **Proposition 17.1.** (Lack of Memory Property)

Let $T = \text{exponential}(\lambda)$, then $\forall\, t, s \geq 0$,

$$\mathbb{P}(T > t+s \mid T > t) = \mathbb{P}(T > s).$$

Pf. $\mathbb{P}(T > t+s \mid T > t)$

$$= \frac{\mathbb{P}(T > t+s,\ T > t)}{\mathbb{P}(T > t)}$$

$$= \frac{\mathbb{P}(T > t+s)}{\mathbb{P}(T > t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = \mathbb{P}(T > s) \qquad \square$$

Interpretation (Waiting bus).

If we've been waiting a bus for t units of time, then the probability we must wait s more units of time is the same as if we haven't waited at all.

Proposition 17.2. (Exponential Races) Let $S = \text{exponential}(\mu)$ and $T = \text{exponential}(\lambda)$ be independent. Let $R = \min\{T, S\}$. Then $R = \text{exponential}(\lambda + \mu)$.

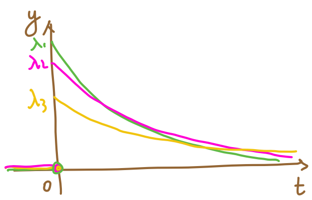

Fig1: PDFs

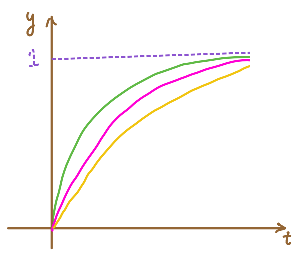

Fig2: CDFs

Pf.

$$
\begin{aligned}
\mathbb{P}(R > t) &= \mathbb{P}(\min\{T, S\} > t) \\
&= \mathbb{P}(T > t, S > t) \\
&= \mathbb{P}(T > t) \cdot \mathbb{P}(S > t) \\
&= e^{-\lambda t} \cdot e^{-\mu t} \\
&= e^{-(\lambda + \mu)t}.
\end{aligned}
$$

This implies that $R = \text{exponential}(\lambda + \mu)$.

Q: What is the probability of "T finishes first"?

A: $\mathbb{P}(T < S)$

$$= \int_0^\infty \mathbb{P}(t < S)\, f_T(t)\, dt$$

$$= \int_0^\infty e^{-\mu t} \cdot \lambda e^{-\lambda t}\, dt$$

$$= \lambda \int_0^\infty e^{-(\mu+\lambda)t}\, dt$$

$$= \lambda \cdot \left. \frac{e^{-(\mu+\lambda)t}}{-(\mu+\lambda)} \right|_0^\infty$$

$$= \frac{\lambda}{\mu+\lambda}.$$

Ex 17.1. Anne and Betty enter a beauty parlor simultaneously, Anne to get a manicure and Betty to get a haircut. Suppose the time for a manicure (resp., haircut) is exponential distributed with mean 20 (resp., 30) mins.

Q(a). What is the probability Anne gets done first?

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A: The rate $\lambda = \frac{1}{20}$, $\mu = \frac{1}{30}$. So

$$\mathbb{P}(\text{Anne finishes first}) = \frac{\lambda}{\lambda+\mu} = \frac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{30}} = \frac{30}{50} = \frac{3}{5}.$$

Q(b). What's the expected amount of time until Anne and Betty are both done?

A: The finish time for the first customer follows exponential $(\lambda+\mu)$. So, $\mathbb{E}$(finish time of first customer)

$$= \frac{1}{\lambda+\mu} = \frac{1}{\frac{1}{20}+\frac{1}{30}} = 12 \text{ mins}.$$

With probability $\frac{3}{5}$, Anne finishes first, then by the Lack of Memory Property, $\mathbb{E}$(finish time of both)

$= \mathbb{E}$(finish time of first customer) $+ \mathbb{E}$(finish time of Betty)

$= 12$ mins $+ 30$ mins $= 42$ mins.

Similarly, with probability $\frac{2}{5}$, $\mathbb{E}$(finish time of both)

$= 12$ mins $+ 20$ mins $= 32$ mins.

Thus, $\mathbb{E}$(finish time of both) $= \frac{3}{5} \times 42 + \frac{2}{5} \times 32 = 38$ mins.

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3°. **Theorem 17.1** (Race of n Exponential RVs)

Let $T_i = \text{exponential}(\lambda_i)$, $1 \le i \le n$, be independent, $V = \min_{1 \le i \le n}\{T_i\}$, and I be the (random) index of the T_i that is smallest. Then

$$\mathbb{P}(V > t) = e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)t},$$

$$\mathbb{P}(I = i) = \frac{\lambda_i}{\lambda_1+\lambda_2+\cdots+\lambda_n}.$$

That is, $V = \text{exponential}(\lambda_1+\lambda_2+\cdots+\lambda_n)$.

Moreover, V and I are independent.

Proof.
$$\begin{aligned}
\mathbb{P}(V>t) &= \mathbb{P}(\min\{T_1, \cdots T_n\} > t) \\
&= \mathbb{P}(T_1 > t, T_2 > t, \cdots, T_n > t) \\
&= \mathbb{P}(T_1 > t) \cdot \mathbb{P}(T_2 > t) \cdot \cdots \cdot \mathbb{P}(T_n > t) \\
&= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot \cdots \cdot e^{-\lambda_n t} \\
&= e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)t}.
\end{aligned}$$

Thus, $V = \text{exponential}(\lambda_1+\lambda_2+\cdots+\lambda_n)$.

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Recall

X_1 & X_2 are independent iff

$F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$,

OR

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

Suppose $I = i$. Let $S = T_i$ and $R = \min_{j \neq i} \{T_j\}$.

Then $R = \text{Exponential}\left(\sum_{j \neq I} \lambda_j\right)$.

Using the same argument above, one has

$$\mathbb{P}(I = i) = \mathbb{P}\left(T_i = \min_j \{T_j\}\right)$$

$$= \mathbb{P}(S < R)$$

$$= \frac{\lambda_i}{\lambda_i + \sum_{j \neq i} \lambda_i} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

Let $f_{i,V}(t)$ be the PDF for V on the set $I = i$.

In order for i to be first at time t, $T_i = t$ and $T_j > t$ $\forall j \neq i$. So

$$f_{i,V}(t) = \lambda_i e^{-\lambda_i t} \cdot \prod_{j \neq i} e^{-\lambda_j t}$$

$$= \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \cdot (\lambda_1 + \cdots + \lambda_n) \cdot e^{-(\lambda_1 + \cdots + \lambda_n)t}$$

$$= \mathbb{P}(I = i) \cdot f_V(t).$$

By definition, V and I are independent. □

Ex 17.2. A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times of Part A, B, C are independent exponential with means 1, 1.5, and 3 years.

Q(a). What's the average length of time the submarine can remain at sea?

A: Let $\lambda = 1$, $\mu = \frac{1}{1.5} = \frac{2}{3}$, $\nu = \frac{1}{3}$. Then

$\mathbb{E}[\text{time before first failure}] = \frac{1}{\lambda+\mu+\nu} = \frac{1}{2}$.

$\mathbb{P}(\text{Part A fails}) = \frac{\lambda}{\lambda+\mu+\nu} = \frac{1}{2}$.

$\mathbb{P}(\text{Part B fails}) = \frac{\mu}{\lambda+\mu+\nu} = \frac{1}{3}$.

$\mathbb{P}(\text{Part C fails}) = \frac{\nu}{\lambda+\mu+\nu} = \frac{1}{6}$.

$\mathbb{E}[\text{time between first and second failure}]$

$= \frac{1}{2} \cdot \frac{1}{\mu+\nu} + \frac{1}{3} \cdot \frac{1}{\lambda+\nu} + \frac{1}{6} \cdot \frac{1}{\lambda+\mu} = \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{6} \cdot \frac{3}{5} = 0.85$.

$\mathbb{E}[\text{time before second failure}] = \frac{1}{2} + 0.85 = 1.35$ years.

Q(b). Find the probability for the six orders in which the failures can occur.

A:

ABC: $\frac{\lambda}{\lambda+\mu+\nu} \cdot \frac{\mu}{\mu+\nu} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

ACB: $\frac{\lambda}{\lambda+\mu+\nu} \cdot \frac{\nu}{\mu+\nu} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

BAC: $\frac{\mu}{\lambda+\mu+\nu} \cdot \frac{\lambda}{\lambda+\nu} = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$.

BCA: $\frac{\mu}{\lambda+\mu+\nu} \cdot \frac{\nu}{\lambda+\nu} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.

CAB: $\frac{\nu}{\lambda+\mu+\nu} \cdot \frac{\lambda}{\lambda+\mu} = \frac{1}{6} \cdot \frac{3}{5} = \frac{1}{10}$

CBA: $\frac{\nu}{\lambda+\mu+\nu} \cdot \frac{\mu}{\lambda+\mu} = \frac{1}{6} \cdot \frac{2}{5} = \frac{1}{15}$.

4°. Theorem 17.2. (Sum of Exponential (λ)s).

Let $\tau_1, \tau_2, \cdots$ be independent exponential (λ). Then the sum $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$ has a gamma (n, λ) distribution. That is, T_n has PDF:

$$f_{T_n}(t) = \begin{cases} \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!}, & t \geq 0; \\ 0, & t < 0. \end{cases}$$

Pf. (Proof by induction).

① For $n=1$, $T_1 = \tau_1 =$ exponential (λ). Then

$$f_{T_1}(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0; \\ 0 & t < 0. \end{cases}$$ is the same as claimed.

② Suppose that the statement is true for n.

For the case $n+1$,

$$f_{T_{n+1}}(t) = f_{T_n + \tau_{n+1}}(t) = \partial_t \left(F_{T_n + \tau_{n+1}}(t) \right).$$

$$F_{T_n + \tau_{n+1}}(t) = \mathbb{P}(T_n + \tau_{n+1} \leq t)$$

$$= \int_0^\infty \mathbb{P}(\tau_{n+1} \leq t - s) f_{T_n}(s) \, ds$$

$$= \int_0^\infty F_{\tau_{n+1}}(t-s) \cdot f_{T_n}(s) \, ds.$$

Therefore,

$$f_{T_{n+1}}(t) = \partial_t F_{T_n + \tau_{n+1}}(t)$$

$$= \partial_t \int_0^\infty F_{\tau_{n+1}}(t-s) \cdot f_{T_n}(s) \, ds$$

$$= \int_0^\infty \partial_t F_{\tau_{n+1}}(t-s) \cdot f_{T_n}(s) \, ds$$

$$= \int_0^\infty f_{\tau_{n+1}}(t-s) \cdot f_{T_n}(s) \, ds$$

By the DCT (Dominated Convergence Theorem)

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$$= \int_0^t \lambda e^{-\lambda(t-s)} \cdot \lambda e^{-\lambda s} \cdot \frac{(\lambda s)^{n-1}}{(n-1)!} ds$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda t} \int_0^t s^{n-1} ds$$

$$= \frac{(\lambda t)^n}{n!} \cdot \lambda e^{-\lambda t} .$$

The statement also holds for $n+1$.

③. By the mathematical induction, this statement holds for all $n \in \mathbb{N}$. $\square$

This is the end of this lecture !